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ON THE GENERALIZED LIE STRUCTURE OF ASSOCIATIVE ALGEBRAS

BY

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Dedicated to the memory of S. A. Amitsur

ABSTRACT

We study the structure of Lie algebras in the category ${}^H\mathcal{M}$ of H-comodules for a cotriangular bialgebra $(H, \langle\ |\ \rangle)$ and in particular the H-Lie structure of an algebra A in ${}^H\mathcal{M}$. We show that if A is a sum of two H-commutative subrings, then the H-commutator ideal of A is nilpotent; thus if A is also semiprime, A is H-commutative. We show an analogous result for arbitrary H-Lie algebras when H is cocommutative. We next discuss the H-Lie ideal structure of A. We show that if A is H-simple

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and H is cocommutative, then any non-commutative H-Lie ideal U of A must contain $[A, A]$. If U is commutative and H is a group algebra, we show that U is in the graded center if A is a graded domain.

Introduction

The generalized Lie algebras considered in this paper are Lie algebras in the category ${}^H\mathcal{M}$ of H-comodules where $(H, \langle\,|\,\rangle)$ is a cotriangular bialgebra over the commutative ring k. They are a special case of the generalized Lie algebras discussed by Gurevich [G] and Manin [Man], and include as special cases Lie superalgebras and Lie coloralgebras. We shall be mostly interested in the H-Lie structure of an associative algebra A in ${}^H\mathcal{M}$; here the definition of the Lie product $[\ ,\ ]$ on A depends on the particular braiding $\langle\,|\,\rangle\colon H \otimes H \to k$ which gives H its cotriangular structure.

We note that Lie algebras in ${}^H\mathcal{M}$ have already been studied in [FM], where a Schur centralizer theorem was proved for $A = \mathrm{End}(V)$, V a finite-dimensional vector space in ${}^H\mathcal{M}$. We also note that if one wishes to study generalized Lie algebras in the category ${}^H\mathcal{M}$ of comodules over any bialgebra H, then in fact H must be cotriangular. For, ${}^H\mathcal{M}$ must be a symmetric monoidal category, and it then follows by [LT] that H is cotriangular; see also [Mo, 10.4.2]. Thus the present setting is the most general possible for generalized Lie algebras of this type.

In Section 2, we study the H-commutativity of A, that is, when $[A, A] = 0$. We show that if A is a sum of two H-commutative subrings, then the H-commutator ideal of A is nilpotent; thus if A is also H-semiprime, A is H-commutative. When H is cocommutative, we obtain an analogous result for any H-Lie algebra $\mathcal{L}$ which is the sum of H-abelian Lie subalgebras. These results generalize work of [BG] for ordinary associative algebras and of [BK] for coloralgebras (the case when $H = kG$, a group algebra).

In Section 3, we turn to the H-Lie ideals of A, and extend some of Herstein's work on the (usual) Lie structure of associative rings [H1], [H2]. We first consider H-Lie ideals U of A for which $[U, U] \neq 0$ and show that the subring of A generated by U contains a non-zero H-ideal of A. If also H is cocommutative and A is H-prime, we show that there exists an H-ideal I of A such that $0 \neq [I, A] \subseteq U$. Thus if A is H-simple, $U \supseteq [A, A]$.

For H-commutative H-Lie ideals U, one would like to show that if A is H-prime, then $U \subseteq Z_H(A)$, the H-center of A, unless A is four-dimensional over k (in which case well-known counter examples already exist, such as the Lie superalgebra $A = \mathrm{gl}(1,1)$). However, the H-commutative case is more difficult, and here we specialize to the case of Lie coloralgebras, that is, $H = kG$. We prove that if A is graded semiprime of characteristic not 2, and U is a graded Lie ideal of A with $[U, U] = 0$, then the even component U_+ of U is contained in the graded center of A (when $H = k\mathbb{Z}_2$, the Lie superalgebra case, this says that the even part of U is central). Moreover, the homogeneous elements of U_-, the odd component, are nilpotent and so $U \subseteq Z_G(A)$ if A is a graded domain.

Finally, in the last section we consider some examples. In particular, the first Weyl algebra $A = \mathbf{A_1}$ is $\mathbb{Z}_2$-graded, and so has the structure of a Lie superalgebra. We apply the results of Section 3 to see that if $U \neq k$ is a (super) Lie ideal of A, then $U \supseteq [A, A]$. We also show that some H-Lie algebras constructed using $H = \mathcal{O}_q(M_n(k))$ can be viewed as Lie coloralgebras for $G = (\mathbb{Z}_2)^n$.

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1. Definitions

Throughout, k denotes a commutative ring, tensoring will be over k, and H will be a bialgebra or a Hopf algebra over k. We use Sweedler's notation [Sw], leaving out subscript parentheses in the summation notation.

Recall that if H is a bialgebra and M a left H-comodule with coaction

$$\rho\colon M \to H \otimes M, \quad m \mapsto \sum m_{-1} \otimes m_0 \quad \forall m \in M, \tag{1.1}$$

then coassociativity of the coaction means $(\Delta \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \rho$; applying this to any $m \in M$ we get

$$\begin{aligned} \sum (m_{-1})_1 \otimes (m_{-1})_2 \otimes m_0 &= \sum m_{-1} \otimes (m_0)_{-1} \otimes (m_0)_0 \\ &= \sum m_{-2} \otimes m_{-1} \otimes m_0. \end{aligned} \tag{1.2}$$

In this paper we consider objects in the category of left H-comodules, ${}^H\mathcal{M}$. In particular, a left H-comodule algebra A is an algebra in this category; this means that multiplication in A is an H-comodule map:

$$\rho(ab) = \rho(a)\rho(b) = \sum a_{-1}b_{-1} \otimes a_0 b_0 \quad \forall a, b \in A. \tag{1.3}$$

Suppose that $\mathcal{C}$ is a symmetric monoidal category [Mac, p.180]; that is, $\mathcal{C}$ has a tensor product on its objects satisfying certain associativity conditions and a twist map

(1.4) $$\tau\colon M \otimes N \to N \otimes M \quad \forall M, N \in \mathcal{C}$$

satisfying the braid conditions, such that $\tau^2 = \mathrm{id}$. Then we may define the concept of a Lie algebra in the category $\mathcal{C}$ as follows:

Definition 1.5: A **Lie algebra $\mathcal{L}$ in the symmetric monoidal category $\mathcal{C}$** is an object $\mathcal{L}$ of $\mathcal{C}$ together with a bracket operation $[\ ,\ ]\colon \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ which is a $\mathcal{C}$-morphism satisfying:

(a) anticommutativity in $\mathcal{C}$: $[\ ,\ ] \circ (\mathrm{id} + \tau) = 0$,

(b) a $\mathcal{C}$-Jacobi identity:

$$[\ ,\ ] \circ ([\ ,\ ] \otimes \mathrm{id})(\mathrm{id} + \tau_{12}\tau_{23} + \tau_{23}\tau_{12}) = 0$$

where τ_{ij} is τ applied to the i and j components of the tensor product $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$.

Gurevich introduced this notion in terms of the "R-matrix" solutions of the quantum Yang–Baxter equation in [G]. Manin more generally defines a "τ-Lie algebra" in [Man], although he does not seem to note that the category must be symmetric, in order that (a) hold.

We are interested in Lie algebras in a particular type of symmetric monoidal category: that of the left H-comodules for a cotriangular H. In this case, ${}^H\mathcal{M}$ being braided monoidal is equivalent to H being coquasitriangular [LT]; adding the symmetry condition $\tau^2 = \mathrm{id}$ forces H to be cotriangular.

Definition 1.6: A pair $(H, \langle\ |\ \rangle)$ is called a **coquasitriangular bialgebra (Hopf algebra)** if H is a bialgebra (Hopf algebra) and $\langle\ |\ \rangle\colon H \otimes H \to k$ is a k-bilinear form satisfying $\forall h, g, \ell \in H$:

(a) $\sum \langle h_1 | g_1 \rangle g_2 h_2 = \sum h_1 g_1 \langle h_2 | g_2 \rangle$,

(b) $\langle\ |\ \rangle$ is convolution invertible in $\mathrm{Hom}_k(H \otimes H, k)$,

(c) $\langle h | g\ell \rangle = \sum \langle h_1 | g \rangle \langle h_2 | \ell \rangle$,

(d) $\langle hg | \ell \rangle = \sum \langle g | \ell_1 \rangle \langle h | \ell_2 \rangle$.

If, in addition, $\langle\ |\ \rangle$ is symmetric, that is

(e) $\sum \langle h_1 | g_1 \rangle \langle g_2 | h_2 \rangle = \varepsilon(g)\varepsilon(h) \quad \forall h, g \in H$,

then $(H, \langle\ |\ \rangle)$ is called **cotriangular**.

The map $\langle\ |\ \rangle$ is called the **braiding**.

In the category ${}^H\mathcal{M}$, the twist map $\tau\colon M\otimes N :\to N\otimes M$, $\quad N, M\in\mathcal{C}$ is given explicitly by

$$\tau(m\otimes n)=\sum\langle m_{-1}|n_{-1}\rangle n_0\otimes m_0 \quad \forall m\in M,\quad n\in N. \tag{1.7}$$

Then symmetry of τ is equivalent to symmetry of the braiding $\langle\ |\ \rangle$.

From here on, assume that $(H,\langle\ |\ \rangle)$ is cotriangular. In this case a Lie algebra $\mathcal{L}$ in ${}^H\mathcal{M}$ is called an ***H*-Lie algebra**, and the conditions in Definition 1.5 can be written explicitly as follows, $\forall n, m, \ell\in\mathcal{L}$:

H-anticommutativity:

$$[m,n]+\sum\langle m_{-1}|n_{-1}\rangle[n_0,m_0]=0, \tag{1.8}$$

H-Jacobi identity:

$$\begin{aligned}[][[\ell,m],n]&+\sum\langle \ell_{-1}m_{-1}|n_{-1}\rangle[[n_0,\ell_0],m_0]\\ &+\sum\langle \ell_{-1}|m_{-1}n_{-1}\rangle[[m_0,n_0],\ell_0]=0.\end{aligned} \tag{1.9}$$

The fact that $[\ ,\ ]$ is a map in ${}^H\mathcal{M}$ means that

$$\rho([m,n])=\sum m_{-1}n_{-1}\otimes[m_0,n_0]. \tag{1.10}$$

Example 1.11: Let A be an algebra in the category $\mathcal{C}={}^H\mathcal{M}$ for some cotriangular bialgebra $(H,\langle\ |\ \rangle)$ as in (1.3). Let A^- be the set A together with the $\mathcal{C}$-Lie bracket $[\ ,\ ]\colon A^-\otimes A^-\to A^-$ defined by

$$[a,b]=ab-\sum\langle a_{-1}|b_{-1}\rangle b_0a_0,$$

$\forall a,b\in A^-$. Then A^- is a $\mathcal{C}$-Lie algebra.

When A is an algebra in ${}^H\mathcal{M}$, we consider some H-analogues of classical concepts of ring theory and of Lie theory. In general, the structures will be H-comodules in addition to the usual requirements.

Definition 1.12: Let A be an algebra in ${}^H\mathcal{M}$.

(a) The ***H*-center** of A is defined to be:

$$Z_H(A):=\{a\in A|\ [a,A]=[A,a]=0\}.$$

(b) A is called ***H*-commutative** if $[A,A]=0$. More generally, $[A,A]$ is the ***H*-commutator** of A.

(c) An ***H*-ideal** I of A is an H-subcomodule of A which is also an ideal of A.

(d) An H-**Lie ideal** U of A is an H-subcomodule of A satisfying $[U, A] \subseteq U$.
(e) A is called H-**prime** if the product of any two nonzero H-ideals of A is nonzero.
(f) A is called H-**semiprime** if A has no nonzero nilpotent H-ideals.
(g) H is called H-**simple** if A has no nontrivial H-ideals.

Example 1.13: If $H = kG$ for some abelian group G with a symmetric bicharacter, then A is a G-graded algebra and the above definitions become the familiar graded ideal, graded prime, and so on [NvO]. However, the graded center depends on the particular choice of bicharacter.

Remark 1.14:

(a) In the definition of H-center of A, a one-sided condition is sufficient if H is a Hopf algebra with bijective antipode, since in this case $\{a \in A | [a, A] = 0\}$ is an H-comodule, as is true in the classical case. This follows from Lemma 3.5(b).

We will see in Corollary 3.6 that $Z_H(A)$ is always a subring of A.

(b) The concept of H-commutative algebras in the dual situation, that is when H is quasitriangular and A is an H-module algebra, was studied in [CW], where they were called "quantum commutative".
(c) Note that $[A, A]$ is an H-comodule because of (1.10), and thus it is an H-Lie ideal of A.
(d) To define an H-Lie ideal, a one-sided condition is sufficient, since $[U, A] \subseteq U \Rightarrow [A, U] \subseteq U$. For, given $u \in U, a \in A, [a, u] = -\sum \langle a_{-1} | u_{-1} \rangle [u_0, a_0]$ by H-anticommutativity; the right side is contained in U since U is an H-subcomodule and $[U, A] \subseteq U$.
(e) The terms H-ideal, H-prime, etc. usually mean that the objects under study are stable under an action of H, rather than a coaction. However, in our situation, any H-comodule is also an H-module, via $h \cdot m = \sum \langle m_{-1} \mid h \rangle m_0$, for any $h \in H$ and $m \in M \in {}^H\mathcal{M}$ with $\rho(m) = \sum m_{-1} \otimes m_0$. Thus our terminology is consistent.

2. Algebras which are sums of H-commutative subalgebras

In this section we consider algebras in ${}^H\mathcal{M}$ which are sums of H-commutative subalgebras in ${}^H\mathcal{M}$. We generalize some recent work of Bahturin and Kegel [BK] for superalgebras, which in turn generalizes work of Bahturin and Giambruno

[BG] for ordinary algebras; both of these papers were inspired by a classical result of Kegel [Kg] which says that a ring which is a sum of two nilpotent subrings is nilpotent.

THEOREM 2.1: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular bialgebra and R an algebra in ${}^H\mathcal{M}$ with A and X subalgebras in ${}^H\mathcal{M}$ which are H-commutative, such that $R = A + X$. Then R satisfies the following identity:*

$$[R, R][R, R] = 0.$$

Proof: It is obvious that it is sufficient to prove the triviality of a product of commutators of the form $[a,x][b,y]$ with $a, b \in A$ and $x, y \in X$. Now it follows from the hypotheses that

$$xb = c + z, \qquad \text{with } c \in A \quad \text{and} \quad z \in X.$$

Writing $\rho(a) = \sum a_{-1} \otimes a_0$ and $\rho(y) = \sum y_{-1} \otimes y_0$, we set

$$a_0 y_0 = f(a_0, y_0) + t(a_0, y_0)$$

for each pair of components a_0, y_0, where $f(a_0, y_0) \in A$ and $t(a_0, y_0) \in X$. Applying $(\mathrm{id} \otimes \rho) \circ \rho = (\Delta \otimes \mathrm{id}) \circ \rho$ to a and to y, we obtain

$$\begin{aligned} &\sum [a_{-1} \otimes y_{-1} \otimes f(a_0, y_0)_{-1} \otimes f(a_0, y_0)_0 \\ &\qquad + a_{-1} \otimes y_{-1} \otimes t(a_0, y_0)_{-1} \otimes t(a_0, y_0)_0] \\ = &\sum [a_{-2} \otimes y_{-2} \otimes a_{-1} y_{-1} \otimes f(a_0, y_0) \\ &\qquad + a_{-2} \otimes y_{-2} \otimes a_{-1} y_{-1} \otimes t(a_0, y_0). \end{aligned} \tag{2.2}$$

We use (2.2) to show an identity we will need below. That is,

$$\begin{aligned} &\sum \langle a_{-1} | x_{-1} \rangle \langle b_{-1} | y_{-1} \rangle x_0 a_0 y_0 b_0 \\ = &\sum \langle a_{-1} | x_{-1} b_{-1} \rangle \varepsilon(y_{-1}) x_0 b_0 f(a_0, y_0) \\ &+ \sum \langle x_{-1} b_{-1} | a_{-1} \rangle \varepsilon(a_{-1}) t(a_0, y_0) x_0 b_0. \end{aligned} \tag{2.3}$$

To see this, we also use the H-commutativity of A and X and the braiding axioms:

$$\begin{aligned}
&\sum\langle a_{-1}|x_{-1}\rangle\langle b_{-1}|y_{-1}\rangle x_0(f(a_0,y_0)+t(a_0,y_0))b_0\\
&=\sum\langle a_{-1}|x_{-1}\rangle\langle b_{-1}|y_{-1}\rangle[x_0\sum\langle f(a_0,y_0)_{-1}|b_{0,-1}\rangle b_{0,0}f(a_0,y_0)_0\\
&\quad+\sum\langle x_{0,-1}|t(a_0,y_0)_{-1}\rangle t(a_0,y_0)_0x_{0,0}b_0]\\
&=\sum\langle a_{-2}|x_{-1}\rangle\langle b_{-1}|y_{-2}\rangle[x_0\sum\langle a_{-1}y_{-1}|b_{0,-1}\rangle b_{0,0}f(a_0,y_0)\\
&\quad+\sum\langle x_{0,-1}|a_{-1}y_{-1}\rangle t(a_0,y_0)x_{0,0}b_0]\quad\text{using (2.2)}\\
&=\sum\langle a_{-2}|x_{-1}\rangle\langle b_{-2}|y_{-2}\rangle\langle a_{-1}y_{-1}|b_{-1}\rangle x_0b_0f(a_0,y_0)\\
&\quad+\sum\langle a_{-2}|x_{-2}\rangle\langle b_{-1}|y_{-2}\rangle\langle x_{-1}|a_{-1}y_{-1}\rangle t(a_0,y_0)x_0b_0\\
&=\sum\langle a_{-2}|x_{-1}\rangle\langle b_{-3}|y_{-2}\rangle\langle y_{-1}|b_{-2}\rangle\langle a_{-1}|b_{-1}\rangle x_0b_0f(a_0,y_0)\\
&\quad+\sum\langle a_{-2}|x_{-3}\rangle\langle b_{-1}|y_{-2}\rangle\langle x_{-2}|a_{-1}\rangle\langle x_{-1}|y_{-1}\rangle t(a_0,y_0)x_0b_0\\
&=\sum\langle a_{-2}|x_{-1}\rangle\langle a_{-1}|b_{-1}\rangle\varepsilon(y_{-1})x_0b_0f(a_0,y_0)\\
&\quad+\sum\langle b_{-1}|y_{-2}\rangle\langle x_{-1}|y_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)x_0b_0\\
&=\sum\langle a_{-1}|x_{-1}b_{-1}\rangle\varepsilon(y_{-1})x_0b_0f(a_0,y_0)\\
&\quad+\sum\langle x_{-1}b_{-1}|y_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)x_0b_0,
\end{aligned}$$

where the last two equalities used Definition 1.6 (c), (d) and (e).

Now we start our computation of $[a,x][b,y]$. We have

$$\begin{aligned}
[a,x][b,y]&=\sum(ax-\langle a_{-1}|x_{-1}\rangle x_0a_0)(by-\langle b_{-1}|y_{-1}\rangle y_0b_0)\\
&=\sum(axby-\langle a_{-1}|x_{-1}\rangle x_0a_0by-\langle b_{-1}|y_{-1}\rangle axy_0b_0\\
&\qquad+\langle a_{-1}|x_{-1}\rangle\langle b_{-1}|y_{-1}\rangle x_0a_0y_0b_0).
\end{aligned}$$

We use H-commutativity of A and X, that $xb=c+x$, and coassociativity of ρ, to rewrite this as:

$$\begin{aligned}
[a,x][b,y]=&acy+azy-\sum\langle a_{-2}|x_{-1}\rangle\langle a_{-1}|b_{-1}\rangle x_0b_0a_0y_0\\
&-\sum\langle b_{-1}|y_{-2}\rangle\langle x_{-1}|y_{-1}\rangle ay_0x_0b_0+\sum\langle a_{-1}|x_{-1}\rangle\langle b_{-1}|y_{-1}\rangle x_0a_0y_0b_0.
\end{aligned}$$

By properties 1.6 (c) and (d) of the braiding, we have for each component

$$\sum\langle a_{-2}|x_{-1}\rangle\langle a_{-1}|b_{-1}\rangle=\langle a_{-1}|x_{-1}b_{-1}\rangle,\qquad\text{and}$$
$$\sum\langle b_{-1}|y_{-2}\rangle\langle x_{-1}|y_{-1}\rangle=\langle x_{-1}b_{-1}|y_{-1}\rangle.$$

Applying these, H-commutativity of A and X, and the fact that $\sum x_{-1}b_{-1}\otimes x_0b_0=\sum c_{-1}\otimes c_0+\sum z_{-1}\otimes z_0$ we continue in the following way:

$$
\begin{aligned}
&[a,x][b,y]\\
&=\sum\langle a_{-1}|c_{-1}\rangle c_0a_0y+\sum\langle z_{-1}|y_{-1}\rangle ay_0z_0-\sum\langle a_{-1}|c_{-1}\rangle c_0a_0y\\
&\quad-\sum\langle a_{-1}|z_{-1}\rangle z_0a_0y-\sum\langle c_{-1}|y_{-1}\rangle ay_0c_0-\sum\langle z_{-1}|y_{-1}\rangle ay_0z_0\\
&\quad+\sum\langle a_{-1}|x_{-1}\rangle\langle b_{-1}|y_{-1}\rangle x_0a_0y_0b_0\\
&=\sum-\langle a_{-1}|z_{-1}\rangle\epsilon(y_{-1})z_0a_0y_0-\sum\langle c_{-1}|y_{-1}\rangle\epsilon(a_{-1})a_0y_0c_0\\
&\quad+\sum\langle a_{-1}|x_{-1}\rangle\langle b_{-1}|y_{-1}\rangle x_0a_0y_0b_0\\
&=\sum-\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0f(a_0,y_0)-\sum\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0t(a_0y_0)\\
&\quad-\sum\langle c_{-1}|y_{-1}\rangle\varepsilon(a_{-1})f(a_0,y_0)c_0-\sum\langle c_{-1}|y_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)c_0\\
&\quad+\sum\langle a_{-1}|x_{-1}b_{-1}\rangle\varepsilon(y_{-1})x_0b_0f(a_0,y_0)\\
&\quad+\sum\langle x_{-1}b_{-1}|a_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)x_0b_0 \text{ (using (2.3) above)}\\
&=-\sum\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0f(a_0,y_0)-\sum\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0t(a_0y_0)\\
&\quad-\sum\langle c_{-1}|y_{-1}\rangle\varepsilon(a_{-1})f(a_0,y_0)c_0-\sum\langle c_{-1}|y_{-1}\rangle\epsilon(a_{-1})t(a_0,y_0)c_0\\
&\quad+\sum\langle a_{-1}|c_{-1}\rangle\varepsilon(y_{-1})c_0f(a_0,y_0)+\sum\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0f(a_0,y_0\\
&\quad+\sum\langle c_{-1}|y_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)c_0+\sum\langle z_{-1}|y_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)z_0\\
&=-\sum\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0t(a_0y_0)-\sum\langle c_{-1}|y_{-1}\rangle\varepsilon(a_{-1})f(a_0,y_0)c_0\\
&\quad+\sum\langle a_{-1}|c_{-1}\rangle\varepsilon(y_{-1})c_0f(a_0,y_0)+\sum\langle z_{-1}|y_{-1}\rangle\varepsilon(a_{-1})t(a_0,y_0)z_0.
\end{aligned}
$$

Next, using (2.2) and Definition 1.6 (c), (d) and (e), we follow the method used in showing (2.3) to show the following:

$$
\begin{aligned}
&\sum[-\langle a_{-1}|z_{-1}\rangle\varepsilon(y_{-1})z_0t(a_0,y_0)+\langle a_{-1}|c_{-1}\rangle\varepsilon(y_{-1})c_0f(a_0,y_0)]\\
&=\sum[-\langle a_{-2}|z_{-2}\rangle\varepsilon(y_{-2})\langle z_{-1}|a_{-1}y_{-1}\rangle t(a_0,y_0)z_0\\
&\qquad+\langle a_{-2}|c_{-2}\rangle\varepsilon(y_{-2})\langle c_{-1}|a_{-1}y_{-1}\rangle f(a_0,y_0)c_0]\\
&=\sum-\langle a_{-2}|z_{-3}\rangle\langle z_{-2}|a_{-1}\rangle\langle z_{-1}|y_{-1}\rangle t(a_0,y_0)z_0\\
&\quad+\sum\langle a_{-2}|c_{-3}\rangle\langle c_{-2}|a_{-1}\rangle\langle c_{-1}|y_{-1}\rangle f(a_0,y_0)c_0\\
&=\sum-\varepsilon(a_{-1})\langle z_{-1}|y_{-1}\rangle t(a_0,y_o)z_0+\sum\varepsilon(a_{-1})\langle c_{-1}|y_{-1}f(a_0,y_0)c_0.
\end{aligned}
$$

Substituting this equality into our previous expression for $[a,x][b,y]$ and cancelling, we see that $[a,x][b,y]=0$. ∎

Recall that the *H-commutator ideal* of R is the ideal generated by $[R, R]$.

COROLLARY 2.4: *Under the hypotheses of the theorem, the H-commutator ideal of R is nilpotent. If R is also H-semiprime, then R is H-commutative.*

Proof: It suffices to show that $[r,s]w[u,v] = 0\ \forall r,s,u,v,w \in R$. But by the definition of the bracket, $[r,s]w = [[r,s],w] + \sum\langle r_{-1}s_{-1}|w_{-1}\rangle w_0[r_0,s_0]$. Now apply the theorem. ∎

When H is cocommutative as well and $\mathcal{L}$ is an H-Lie algebra, we prove an identity for the elements of $\mathcal{L}$. This includes the case when $\mathcal{L}$ is a Lie coloralgebra, for then $H = kG$ is cocommutative, and thus extends [BK].

We call $\mathcal{L}$ *H-abelian* if $[\mathcal{L},\mathcal{L}] = 0$.

THEOREM 2.5: *Let $\mathcal{L}$ be a Lie algebra in the category ${}^H\mathcal{M}$ where H is a cotriangular cocommutative Hopf algebra. Suppose $\mathcal{L} = A + Z$ where A and Z are Lie subalgebras in ${}^H\mathcal{M}$ which are H-abelian. Then we have*

$$[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]] = 0.$$

Proof: It suffices to show that $[[a,x],[b,y]] = 0$ holds for $a,b \in A$ and $x,y \in X$. Now by (1.9),

$$\begin{aligned}
[[a,x],[b,y]] =& \sum -\langle a_{-1}x_{-1}|b_{-1}y_{-1}\rangle\,[[[b_0,y_0],a_0],x_0] \\
&- \sum\langle a_{-1}|x_{-1}b_{-1}y_{-1}\rangle[[x_0,[b_0,y_0]],a_0] \\
=& \sum\langle a_{-2}x_{-1}|b_{-2}y_{-2}\rangle\{\langle b_{-1}y_{-1}|a_{-1}\rangle[[[a_0,b_0],y_0],x_0] \\
&+ \langle b_{-1}|y_{-1}a_{-1}\rangle[[[y_0,a_0],b_0],x_0]\} \\
&+ \sum\langle a_{-1}|x_{-2}b_{-2}y_{-2}\rangle\langle x_{-1}|b_{-1}y_{-1}\rangle[[[b_0,y_0],x_0],a_0] \\
=& \sum\langle a_{-2}x_{-1}|b_{-2}y_{-2}\rangle\langle b_{-1}|y_{-1}a_{-1}\rangle[[[y_0,a_0],b_0]x_0] \\
&- \sum\langle a_{-1}|x_{-3}b_{-3}y_{-3}\rangle\langle x_{-2}|b_{-2}y_{-2}\rangle \\
&\qquad \times \{\langle b_{-1}y_{-1}|x_{-1}\rangle[[[x_0,b_0],y_0],a_0] + 0\}.
\end{aligned}$$

For each term $[y_0,a_0]$ and $[x_0,b_0]$ we may substitute

$$[y_0,a_0] = c(y_0,a_0) + z(y_0,a_0) \text{ and } [x_0,b_0] = d(x_0,b_0) + w(x_0,b_0)$$

where $c, d \in A$ and $z, w \in Z$. It follows that

$$
\begin{aligned}
[[a,x],[b,y]] &= \sum \langle a_{-2}x_{-1}|b_{-2}y_{-2}\rangle\langle b_{-1}|y_{-1}a_{-1}\rangle[[z,b_0],x_0] \\
&\quad - \sum \langle a_{-1}|x_{-1}b_{-1}y_{-1}\rangle[[d,y_0],a_0] \\
&= \sum -\langle a_{-2}x_{-2}|b_{-3}y_{-2}\rangle\langle b_{-2}|y_{-1}a_{-1}\rangle \\
&\qquad \times \{0 + \sum \langle z_{-1}|b_{-1}x_{-1}\rangle[[b_0,x_0],z_0]\} \\
&\quad + \sum \langle a_{-1}|x_{-1}b_{-1}y_{-2}\rangle\{0 + \langle d_{-1}|y_{-1}a_{-1}\rangle[[y_0,a_0],d_0]\} \\
&= \sum -\langle a_{-3}x_{-2}|b_{-3}y_{-3}\rangle \\
&\qquad \langle b_{-2}|y_{-2}a_{-2}\rangle\langle y_{-1}a_{-1}|b_{-1}x_{-1}\rangle[[b_0,x_0],z_0] + 2^{nd}\text{term}
\end{aligned}
$$

$$
\left(\text{since } \rho([y,a]) = \sum y_{-1}a_{-1} \otimes [y_0,a_0] = \sum c_{-1} \otimes c_0 + \sum z_{-1} \otimes z_0\right)
$$

$$
\begin{aligned}
&= \sum -\langle a_{-2}x_{-2}|b_{-1}y_{-2}\rangle\langle y_{-1}a_{-1}|x_{-1}\rangle[[b_0,x_0],z_0] + 2^{nd}\text{ term} \\
&= \sum \langle a_{-2}x_{-3}|b_{-2}y_{-2}\rangle\langle y_{-1}a_{-1}|x_{-2}\rangle\langle b_{-1}|x_{-1}\rangle[[x_0,b_0],z_0] \\
&\quad + \sum \langle a_{-1}|x_{-2}b_{-2}y_{-2}\rangle\langle x_{-1}b_{-1}|y_{-1}a_{-1}\rangle[[y_0,a_0],d_0]
\end{aligned}
$$

$$
\left(\text{since } \rho([x,b]) = \sum x_{-1}b_{-1} \otimes [x_0,b_0] = \sum d_{-1} \otimes d_0 + \sum w_{-1} \otimes w_0\right)
$$

$$
\begin{aligned}
&= \sum \langle a_{-2}x_{-3}|b_{-2}y_{-2}\rangle\langle y_{-1}a_{-1}|x_{-2}\rangle\langle b_{-1}|x_{-1}\rangle[d,z_0] \\
&\quad + \sum \langle a_{-1}|x_{-2}b_{-2}y_{-2}\rangle\langle x_{-1}b_{-1}|y_{-1}a_{-1}\rangle[z,d_0] \\
&= \sum \langle a_{-2}x_{-3}|b_{-2}y_{-2}\rangle\langle y_{-1}a_{-1}|x_{-2}\rangle\langle b_{-1}|x_{-1}\rangle[d,z] \\
&\quad + \sum \langle a_{-1}|x_{-2}b_{-2}y_{-2}\rangle\langle x_{-1}b_{-1}|y_{-1}a_{-1}\rangle[z,d] \\
&= -\sum \langle a_{-3}x_{-3}|b_{-2}y_{-2}\rangle\langle y_{-1}a_{-1}|x_{-2}\rangle \\
&\quad \langle b_{-1}|x_{-1}\rangle\langle d_{-1}|z_{-1}\rangle[z_0,d_0] + 2^{nd}\text{ term} \\
&= -\sum \langle a_{-3}x_{-4}|b_{-3}y_{-3}\rangle\langle y_{-2}a_{-2}|x_{-3}\rangle\langle b_{-2}|x_{-2}\rangle \\
&\qquad \times \langle x_{-1}b_{-1}|y_{-1}a_{-1}\rangle[z,d] + 2^{nd}\text{term.}
\end{aligned}
$$

Now we need to compare the terms $\langle a_{-1}|x_{-2}b_{-2}y_{-2}\rangle$ and $\langle a_{-3}x_{-4}|b_{-3}y_{-3}\rangle\langle y_{-2}a_{-2}|x_{-3}\rangle\langle b_{-2}|x_{-2}\rangle$:

$$\begin{aligned}
&\sum\langle a_{-3}x_{-3}|b_{-2}y_{-2}\rangle\langle y_{-2}a_{-2}|x_{-3}\rangle\langle b_{-2}|x_{-2}\rangle\\
&=\sum\langle a_{-4}x_{-5}|b_{-3}\rangle\langle a_{-4}x_{-4}|y_{-3}\rangle\langle y_{-2}a_{-2}|x_{-3}\rangle\langle b_{-2}|x_{-2}\rangle\\
&=\sum\langle a_{-4}x_{-6}|b_{-3}\rangle\langle a_{-4}x_{-5}|y_{-3}\rangle\langle y_{-2}|x_{-3}\rangle\langle a_{-2}|x_{-4}\rangle\langle b_{-2}|x_{-2}\rangle\\
&=\sum\langle a_4|b_{-3}\rangle\langle x_{-5}|b_{-4}\rangle\langle a_{-4}|y_{-3}\rangle\langle x_{-5}|y_{-4}\rangle\langle y_{-2}|x_{-3}\rangle\langle a_{-2}|x_{-4}\rangle\langle b_{-2}|x_{-2}\rangle\\
&=\sum\langle a_{-3}|x_{-2}\rangle\langle a_{-2}|b_{-2}\rangle\langle a_{-1}|y_{-2}\rangle \qquad \text{(since } H \text{ is cocommutative)}\\
&=\sum\langle a_{-1}|x_{-2}b_{-2}y_{-2}\rangle.
\end{aligned}$$

Thus the whole sum is zero, as required. ∎

3. On the H-Lie ideal structure of A

We first recall Herstein's results [H1], [H2], which generalized the classical facts about Lie ideals in matrix rings. He proved that if A is any simple ring, considered as a Lie algebra under the usual [,], and U is a Lie ideal of A, then either $U \supseteq [A,A]$ or $U \subseteq Z(A)$, the center of A, unless A has characteristic 2 and is four-dimensional over $Z(A)$. It is this result which we would like to extend to the case of H-simple algebras.

However, we note that for H-algebras the four-dimensional case will be an exception, in any characteristic. For, the Lie superalgebra $A = \mathrm{gl}(1,1)$ has a non-central Lie ideal U properly contained in $[A,A] = \mathrm{sl}(1,1)$; see 4.2. We conjecture that for any cotriangular Hopf algebra H and H-simple algebra A in ${}^H\mathcal{M}$, any H-Lie ideal U of A must either contain $[A,A]$ or be contained in $Z_H(A)$, unless A is 4-dimensional.

Although unable to prove this in general, we make some progress. The following lemma is used frequently.

LEMMA 3.1: *Let A be an algebra in ${}^H\mathcal{M}$ and let $m,n,\ell \in A$. Then*

(a) $[m,n\ell] = [m,n]\ell + \sum\langle m_{-1}|n_{-1}\rangle n_0[m_0,\ell]$,

(b) $[mn,\ell] = m[n,\ell] + \sum\langle n_{-1}|\ell_{-1}\rangle[m,\ell_0]n_0$.

If H is also cocommutative, then

(c) $[mn,\ell] = [m,n\ell] + \sum\langle m_{-1}|n_{-1}\ell_{-1}\rangle[n_0,\ell_0 m_0]$.

Proof: (a) $[m,n]\ell$

$$
\begin{aligned}
&= mn\ell - \sum \langle m_{-1}|n_{-1}\rangle n_0 m_0 \ell \\
&= mn\ell - \sum \langle m_{-1}|n_{-1}\ell_{-1}\rangle n_0\ell_0 m_0 + \sum \langle m_{-1}|n_{-1}\ell_{-1}\rangle n_0 \ell_0 m_0 \\
&\quad - \sum \langle m_{-1}|n_{-1}\rangle n_0 m_0 \ell \\
&= [m, n\ell] - (\sum \langle m_{-1}|n_{-1}\rangle n_0 m_0 \ell - \sum \langle m_{-2}|n_{-1}\rangle \langle m_{-1}|\ell_{-1}\rangle n_0 \ell_0 m_0) \\
&\quad \text{(by property 1.6c)} \\
&= [m, n\ell] - \sum_{m,n} \langle m_{-1}|n_{-1}\rangle n_0 \sum_{m_0} (m_0 \ell - \sum \langle (m_0)_{-1}|\ell_{-1}\rangle \ell_0 (m_0)_0) \\
&\quad \text{(by coassociativity)} \\
&= [m, n\ell] - \sum \langle m_{-1}|n_{-1}\rangle n_0 [m_0, \ell].
\end{aligned}
$$

(b) This is similar to (a).

(c) $[m, n\ell] + \sum \langle m_{-1}|n_{-1}\ell_{-1}\rangle [n_0, \ell_0 m_0]$

$$
\begin{aligned}
&= mn\ell - \sum \langle m_{-1}|n_{-1}\ell_{-1}\rangle n_0 \ell_0 m_0 + \sum \langle m_{-1}|n_{-1}\ell_{-1}\rangle n_0 \ell_0 m_0 \\
&\quad - \sum \langle m_{-1}|n_{-1}\ell_{-1}\rangle \langle (n_0)_{-1}|(\ell_0)_{-1}(m_0)_{-1}\rangle (\ell_0)_0 (m_0)_0 (n_0)_0 \\
&= mn\ell - \sum \langle m_{-3}|n_3\rangle \langle m_{-2}|\ell_{-2}\rangle \langle n_{-2}|\ell_{-1}\rangle \langle n_{-1}|m_{-1}\rangle \ell_0 m_0 n_0 \\
&\quad \text{(by coassociativity and properties of the braiding)} \\
&= mn\ell - \sum \langle m_{-3}|n_{-3}\rangle \langle n_{-2}|m_{-2}\rangle \langle m_{-1}|\ell_{-1}\rangle \langle n_{-1}|\ell_{-2}\rangle \ell_0 m_0 n_0 \\
&\quad \text{(since } H \text{ is cocommutative)} \\
&= mn\ell - \sum \langle m_{-1}n_{-1}|\ell_{-1}\rangle \ell_0 m_0 n_0 = [mn, \ell]. \quad \blacksquare
\end{aligned}
$$

Remark 3.2: Part (a) of the lemma essentially says that $d\colon A \to A$ given by $d(a) = [m, a]$ is a derivation in the category ${}^H\mathcal{M}$. For, a derivation d would have to satisfy

$$d \cdot (a \otimes b) = (d \otimes 1) \cdot (a \otimes b) + (1 \otimes d) \cdot (a \otimes b) \qquad \text{for all } a, b \in A$$

and the fact that d is a derivation in a symmetric monoidal category means we must use the twist map to carry this out.

Our first result holds for any bialgebra H. It is a replacement for [H2, Lemma 1.4] in which a different set $T(U)$ was used. See the remarks after Corollary 3.9.

LEMMA 3.3: *Let U be an H-Lie ideal of A, and let $S(U)$ be the subring generated by U. Then:*

(a) *$S(U)$ is an H-Lie ideal of A,*

(b) *if also H is cocommutative, then $[S(U), A] \subseteq U$.*

Proof: (a) $S(U)$ is an H-comodule by (1.3). To see that it is a Lie ideal, we show that $[U^n, A] \subseteq U^{n+1}$ for all $n \leq 1$. Assume true for $n-1$, and choose $x_1, \dots, x_n \in U$, $\quad a \in A$. Then by 3.1(b),

$$\begin{aligned}[x_1 \cdots x_{n-1} x_n, a] &= x_1 \cdots x_{n-1}[x_n, a] + \sum \langle (x_n)_{-1} | a_{-1} \rangle [x_1 \cdots x_{n-1}, a_0](x_n)_0 \\ &\in U^{n-1}[U, A] + [U^{n-1}, A]U \subseteq U^n U \subseteq U^{n+1}.\end{aligned}$$

Thus $S(U)$ is also an H-Lie ideal.

(b) Again by induction, we show $[U^n, A] \subseteq U$. Since H is cocommutative we may use Lemma 3.1(c). As before, choose $x_1, \dots, x_n \in U$ and $a \in A$. Then

$$\begin{aligned}[x_1 \cdots x_{n-1} x_n, a] =& [x_1 \cdots x_{n-1}, x_n a] \\ &+ \sum \langle (x_1 \cdots x_{n-1})_{-1} | (x_n a)_{-1} \rangle [(x_n)_0, (a x_1 \cdots x_{n-1})_0] \\ \in& [U^{n-1}, A] + [U, A] \subseteq [U, A]\end{aligned}$$

by induction. Thus $[S(U), A] \subseteq U$. ∎

Proposition 3.4: *Let A be an algebra in ${}^H\mathcal{M}$ and assume that U is an H-Lie ideal of A such that $[U, U] \neq 0$. Then the subring $S(U)$ generated by U contains a nonzero H-ideal of A.*

Proof: By Lemma 3.3, replacing U by $S(U)$, we may assume that U is also a subring. Choose $u, w \in U$ with $[u, w] \neq 0$. For any $a \in A$, by Lemma 3.1(a) we have

$$(*) \qquad [u, w]a = [u, wa] - \sum \langle u_{-1} | w_{-1} \rangle w_0 [u_0, a]$$

which is in U since U is an H-comodule, a Lie ideal, and a subring.

Now $\forall b \in A, [b, [u, w]a] \in U$ by Remark 1.14(c). Also

$$b[u, w]a = [b, [u, w]a] + \sum \langle b_{-1} | u_{-1} w_{-1} a_{-1} \rangle [u_0, w_0] a_0 b_0.$$

Since U is an H-comodule and $u, w \in U$, it follows that all $u_0, w_0 \in U$, and thus $[u_0, w_0] a_0 b_0 \in U$ by $(*)$. Thus $b[u, w]a \in U$, and so $I := A[U, U]A \subseteq U$. I is nonzero since $1 \in A$ and $[U, U] \neq 0$. ∎

In fact, Proposition 3.5 is true even if A does not have a unit element, since we may use the ideal

$$I = W + AW + WA + AWA \neq 0$$

where $W = [U, U]$.

In the following results, H will be a Hopf algebra with bijective antipode S; the inverse of S is denoted $\bar{S}$.

The proof of the following lemma is obvious, using (1.10) for part (b).

LEMMA 3.5: *If H is a Hopf algebra with bijective antipode, then the following hold for all $x, a \in A$:*

(a) *If A is an H-comodule algebra, then*

$$\sum x_{-1} \otimes x_0 a = \sum \rho(x a_0)(\bar{S}(a_{-1}) \otimes 1).$$

(b) *If A is an H-Lie algebra, then*

$$\sum x_{-1} \otimes [x_0, a] = \sum \rho([x, a_0])(\bar{S}(a_{-1}) \otimes 1).$$

Note the analogy to the well known formula for A an H-module algebra: $h \cdot a = \sum h_2(\bar{S}(h_1) \cdot a)$ $\forall a \in A, h \in H$.

COROLLARY 3.6: *The H-center of A, $Z_H(A)$, is an H-subcomodule algebra.*

Proof: Let $a \in A$, $r, s \in Z_H(A)$. Then by Lemma 3.1(a),

$$[a, rs] = [a, r]s - \sum \langle a_{-1} | r_{-1} \rangle r_0 [a_0, s] = 0$$

since r and s are in the H-center of A. Thus $Z_H(A)$ is a subalgebra.

The fact that $Z_H(A)$ is a subcomodule follows from Lemma 3.5(b); let $a \in Z_H(A)$ and $x \in A$. Then $\sum a_{-1} \otimes [a_0, x] = \sum \rho([a, x_0])(\overline{S}(x_{-1}) \otimes 1) = 0$. Now taking the summands $\{a_{-1}\}$ to be linearly independent, we have $[a_0, x] = 0$ for each summand a_0. ∎

LEMMA 3.7: *If H is a Hopf algebra with bijective antipode, then the annihilator of an H-ideal of A is an H-ideal.*

Proof: Let I be an H-ideal and X its annihilator. Let $x \in X$ and $z \in I$. Then by Lemma 3.5(a), $\sum x_{-1} \otimes x_0 z = \rho(x z_0)(\bar{S}(z_{-1}) \otimes 1) = 0$ since I is a subcomodule of A. This fact is sufficient to show that X is an H-comodule. For, we may choose the $\{x_{-1}\}$ components of $\rho(x)$ to be linearly independent and then each $x_0 z = 0$, implying that each x_0 component is in X. Also, X is clearly an ideal. It follows that X is an H-ideal. ∎

THEOREM 3.8: *Let H be a cocommutative Hopf algebra, let A be an H-prime H-comodule algebra and let U be an H-Lie ideal of A such that $[U,U] \neq 0$. Then there exists an H-ideal I of A such that $0 \neq [I,A] \subseteq U$.*

Proof: Consider $S(U)$, the subring generated by U. Since $[U,U] \neq 0$, $S(U)$ contains a nonzero H-ideal of A, say I, by Proposition 3.4. By Lemma 3.3, $I \subseteq S(U)$ implies that $[I,A] \subseteq U$; that $[I,A] \neq 0$ we see as follows:

Suppose $[I,A] = 0$ and choose $x \in I$. Then

$$x[a,b] = [xa,b] - \sum \langle a_{-1}|b_{-1}\rangle [x,b_0]a_0$$

by Lemma 3.1(b), and this equals 0 since $xa \in I$ and $[I,A] = 0$. But then $I[A,A] = 0$, giving $[A,A] \subseteq \mathrm{Ann}_A(I)$. However, $\mathrm{Ann}_A(I)$ is an H-ideal since I is one, by Lemma 3.7, and A is H-prime, so this implies $[A,A] = 0$, contradicting $[U,U] \neq 0$. ∎

COROLLARY 3.9: *Let H be a cocommutative Hopf algebra and let A be an H-simple algebra in ${}^H\mathcal{M}$. If U is an H-Lie ideal of A with $[U,U] \neq 0$, then $U \supseteq [A,A]$.*

We do not know whether the hypothesis that H is cocommutative is needed in Theorem 3.8; however the method of proof does not work otherwise, since Proposition 3.4 depends on Lemma 3.1(c), which requires cocommutativity. In the classical case, Herstein uses the set $T(U) = \{a \in A | [a,A] \subset U\}$ rather than our set $S(U)$; however his arguments also use a version of Lemma 3.1(c) [H1]. In our case the set $T(U)$ can be shown to be an H-Lie ideal and subring of A.

All the above results apply to the special case of $H = kG$ for an abelian group G with a symmetric bicharacter. For then H is a cocommutative Hopf algebra (and so has a bijective antipode) and is cotriangular. The H-comodule structures of a G-graded algebra $A = \bigoplus_{g \in G} A_g$ are the G-graded ones, for example G-graded ideals.

When $H = kG$ as above, we can extend our results to investigate the situation of $[U,U] = 0$. When G is trivial, our arguments here reduce to those of Herstein for usual Lie algebras.

Definition 3.10: Let G be an abelian group with a symmetric bicharacter $\langle\ |\ \rangle$.

(a) Define $G_+ := \{g \in G|\ \langle g|g\rangle = 1\}$ and $G_- := \{g \in G|\ \langle g|g\rangle = -1\}$.

Note that these are the only possibilities for g since symmetry of $\langle\ |\ \rangle$ implies $\langle g|g\rangle^2 = 1, \quad \forall g \in G$.

(b) For A a G-graded algebra, define

$$A_+ := \bigoplus_{g\in G_+} A_g \quad \text{and} \quad A_- := \bigoplus_{g\in G_-} A_g.$$

LEMMA 3.11: *Assume A is G-graded semiprime of characteristic $\neq 2$, and $a \in A$ is homogeneous such that $[a,[a,A]] = 0$. Then:*

(a) *if $a \in A_+$ then $[a,A] = 0$ (thus $a \in Z_G(A)$),*

(b) *if $a \in A_-$ then $[a^2,A] = 0$ (thus $a^2 \in Z_G(A)$).*

Proof: (a) Say $a \in A_g, r \in A_h, s \in A_\ell$. By Lemma 3.1(a), $[a,rs] = [a,r]s + \langle g|h\rangle r[a,s]$. Thus

$$\begin{aligned} 0 =&[a,[a,rs]] = [a,[a,r]s] + \langle g|h\rangle[a,r[a,s]] \\ =& ([a,[a,r]]s + \langle g|ghl\rangle[a,r][a,s]) + \langle g|h\rangle\left([a,r][a,s] + \langle g|ghl\rangle r[a,[a,s]]\right) \\ =&\langle g|h\rangle\left(1 + \langle g|g\rangle\right)[a,r][a,s]. \end{aligned}$$

Replacing s by sr, and using the fact that $[a,sr] = [a,s]r + \langle g|h\rangle s[a,r]$, we get

$$0 = [r,a][a,sr] = [r,a]s[a,r]$$

and so by graded anticommutativity $0 = [r,a]A[a,r]$. A graded semiprime implies $[r,a] = 0$ for any homogeneous $r \in A$. But then $[a,A] = 0$, or $a \in Z_G$.

(b) This part does not need char $\neq 2$. For $r \in A_h, a \in A_g, \langle g|g\rangle = -1$:

$$\begin{aligned} 0 =&[a,[a,r]] = a[a,r] - \langle g|gh\rangle[a,r]a \\ =&a(ar - \langle g|h\rangle ra) - \langle g|gh\rangle(ar - \langle g|h\rangle ra)a \\ =&a^2r - \langle g|h\rangle ara - \langle g|gh\rangle ara + \langle g|gh\rangle\langle g|h\rangle ra^2 \\ =&a^2r - \langle g^2|h\rangle ra^2 = [a^2,r]. \end{aligned}$$

Thus $[a^2,A] = 0$. ∎

COROLLARY 3.12: *Let A be graded semiprime of* char $\neq 2$, *and let U be a nonzero G-Lie ideal of A such that $[U,U] = 0$. Then $U_+ \subseteq Z_G$, and $a^2 = 0$ for all homogeneous elements a of U_-.*

Proof: Since $[U,U] = 0$, $[a,[a,A]] = 0\ \forall a \in U$. The statement now follows from Lemma 3.11 and the fact that $[a,a] = (1 - \langle g|g\rangle)a^2 = 0$. ∎

Recall that a G-graded algebra A is **graded simple** if it has no non-trivial graded ideals, and is a **graded domain** if it has no homogeneous zero divisors.

COROLLARY 3.13: *Let A be a graded simple graded domain of characteristic not 2. Fix a bicharacter $\langle\,|\,\rangle$ on G and consider A^- in ${}^{kG}\mathcal{M}$ as a Lie coloralgebra using $\langle\,|\,\rangle$. If U is any Lie ideal of A, then either $U \supseteq [A,A]$ or $U = U_+ \subseteq Z_G(A)$, the graded center of A.*

Proof: If $[U,U] \neq 0$, then $U \supseteq [A,A]$ by Corollary 3.9. Thus we may assume $[U,U] = 0$. By Corollary 3.12, $U_- = 0$ since A is a graded domain. Thus $U = U_+ \subseteq Z_G$, and we are done. ∎

4. Examples

Example 4.1: Let $A = \mathbf{A_1}$, the first Weyl algebra. Writing

$$\mathbf{A_1} = k\langle x, y | xy - yx = 1\rangle,$$

it is $\mathbb{Z}_2$-graded by setting $(\mathbf{A_1})_1$ = span of odd-degree monomials and $(\mathbf{A_1})_0$ = span of even-degree monomials. Then $\mathbf{A_1}^-$ becomes a Lie superalgebra in the usual way. We claim that any $\mathbb{Z}_2$-graded Lie ideal $U \neq k \cdot 1$ of A must contain $[A,A]$. For if not, $U = U_0$ is contained in the graded center of A by Corollary 3.13. But the even part of the graded center is contained in the usual center, which is k.

We now give an example of a non-central Lie ideal.

Example 4.2: Let k be a field of characteristic $\neq 2$. We may express $A = \mathrm{gl}(1,1)$ more concretely as follows. Let $A = M_2(k)$ be $\mathbb{Z}_2$-graded, with

$$A_0 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

Then A^- is a Lie superalgebra under the usual superbracket; we have $\langle g|g\rangle = -1$ if $\mathbb{Z}_2 = \langle g\rangle$. One can check here that $Z_G(A) = \{aI | a \in k\}$, the usual center, and that $[A,A] = \left\{\begin{bmatrix} a & b \\ c & a \end{bmatrix}\right\}$ (these have "supertrace" 0). Let $U = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right\}$; then $[U,U] = 0$, $[U,A] \subseteq U$, but U is not graded central and $[A,A] \not\subseteq U$. However, $U_+ = U_0 \subseteq Z_G(A)$, as predicted by the corollary.

This example is really not new, as it is well-known that $\mathrm{sl}(1,1)$ is nilpotent. More generally, it is known that if $A = \mathrm{gl}(n,m)$, then $\mathrm{sl}(n,m) = [A,A]$ is a simple Lie superalgebra if $n \neq m$, and $\mathrm{sl}(n,n)/Z$ is simple when $n \neq 1$, where Z is the scalar matrices [FrKp],[Kc]. Moreover, Herstein actually proves that if

A is any simple ring, then $[A,A]/[A,A]\cap Z$ is a simple Lie algebra unless A is four-dimensional over Z and A has characteristic 2. Thus there may be some hope of showing that in general, if A is H-simple, then $[A,A]/[A,A]\cap Z_H$ is a simple H-Lie algebra except for some low-dimensional cases.

It does not seem to be easy to give examples of algebras A in ${}^H\mathcal{M}$ such that the H-Lie algebra A^- cannot also be described as a G-Lie coloralgebra for some group G for which A is a G-graded algebra. In fact many of the known examples have this property; in particular we show this for examples in ${}^H\mathcal{M}$ when $H = \mathcal{O}_q(M_n(k))$ is cotriangular. We use the Fadeev–Reshetikhin–Takhtadjan construction of $\mathcal{O}_q(M_n(k))$ as formulated in [LT] and [Sm]. Note that in order to form the generalized Lie algebras we need a symmetric category, and so H must be cotriangular. This in turn necessitates that the braiding be symmetric, i.e. $R^2 = I$, hence that $q^2 = 1$.

Example 4.3: Let k be a field of characteristic $\neq 2$ and let $H = \mathcal{O}_q(M_2(k))$ with $q = -1$. Then the following hold:

(1) $H = \mathcal{O}_R(M_n(k)) = k\langle t_i^j\rangle/I_R$, where $i,j \in \{1,\dots,n\}$, for

$$R = -\sum e_{ii}^{ii} + \sum_{i\neq j} e_{ij}^{ij} \quad \text{and} \quad B = \tau\circ R = -\sum e_{ii}^{ii} + \sum_{i\neq j} e_{ij}^{ji}$$

where e_{ij}^{kl} is the $n^2\times n^2$ matrix with 1 in the ij-row and kl-column (the rows and columns are numbered lexicographically). I_R is the ideal of relations in $k\langle t_i^j\rangle$ determined by $B = \tau\circ R$ (see [Sm, pp. 155–158]).

For example, if $n = 2$ then

$$R = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \text{and so} \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Thus, setting $T_i^j = t_i^j + I_R$, H has generators T_i^j with relations as follows: for any 2×2 "submatrix" $\begin{bmatrix} T_i^k & T_i^m \\ T_j^k & T_j^m \end{bmatrix}$ (where $i<j$ and $k<m$), adjacent entries (horizontally or vertically) anticommute, whereas diagonally opposite elements commute.

H is a bialgebra by setting

$$\triangle T_i^j = \sum_{k=1}^{n} T_i^k \otimes T_k^j \quad \text{and} \quad \varepsilon(T_i^j) = \delta_{ij}.$$

(2) H has a braiding as in [LT] given by $\langle T_i^k \mid T_j^\ell \rangle = B_{ij}^{\ell k}$. Explicitly, this says in our case:

$$\langle T_i^i \mid T_i^i \rangle = -1 \quad \text{and} \quad \langle T_i^i \mid T_j^j \rangle = 1 \ \forall i \neq j,$$

and for all other pairs of generators, $\langle T_i^k \mid T_j^\ell \rangle = 0$. Since $R^2 = I$, the braiding is symmetric and H is cotriangular.

(3) $A = H$ is an H-comodule algebra as usual, by taking $\rho = \triangle$. Thus we may consider H^- as an H-Lie algebra. Using part (2) and Example 1.11, we compute the bracket [,] on generators:

$$\begin{aligned}[T_i^k, T_j^l] &= T_i^k T_j^\ell - \sum_{m,n} \langle T_i^m \mid T_j^n \rangle T_n^\ell T_m^k \\ &= T_i^k T_j^\ell - \langle T_i^i \mid T_j^j \rangle T_j^\ell T_i^k \\ &= \begin{cases} T_i^k T_i^\ell + T_i^\ell T_i^k & \text{if } i = j, \\ T_i^k T_j^\ell - T_j^\ell T_i^k & \text{if } i \neq j. \end{cases}\end{aligned}$$

In particular, $[T_i^i, T_i^i] = 2(T_i^i)^2$ and $[T_i^i, T_j^j] = 0$.

Proposition 4.4: *Consider $A^- = H^-$ with [,] as above.*

(a) *If $n > 1$, it is not possible to give H a $\mathbb{Z}_2$-grading such that [,] is the Lie superbracket.*

(b) *For $G = (\mathbb{Z}_2)^n$, H is a G-graded algebra such that the G-Lie bracket coincides with [,] as above.*

Proof: (a) First, assume H is $\mathbb{Z}_2$-graded, say $H = H_0 \oplus H_1$, such that H^- is a Lie superalgebra under [,]. Let $T_i^i = z_0 + z_1$, with $z_0 \in (H^-)_0$, $z_1 \in (H^-)_1$. Then $[T_i^i, T_i^i] = [z_0 + z_1, z_0 + z_1] = [z_0, z_0] + [z_0, z_1] + [z_1, z_0] + [z_1, z_1] = 2z_1^2$. For, the bracket on even elements is the usual one, by graded anticommutativity $[z_0, z_1] = -(-1)^{0\cdot 1}[z_1, z_0] = -[z_1, z_0]$, and $[z_1, z_1] = z_1^2 + z_1^2 = 2z_1^2$ since z_1 is odd. By Example 4.3 (3), $[T_i^i, T_i^i] = 2(T_i^i)^2 = 2(z_0^2 + z_0 z_1 + z_1 z_0 + z_1^2)$. Thus $z_1^2 = z_0^2 + z_0 z_1 + z_1 z_0 + z_1^2$; comparing even and odd components gives $z_1^2 = x_0^2 + z_1^2$, so $z_0 = 0$ since H is a domain. Hence each T_i^i must be odd. But for $i \neq j$, $[T_i^i, T_j^j] = T_i^i T_j^j - T_j^j T_i^i$, contradicting T_i^i and T_j^j odd.

(b) Consider $G = k(\mathbb{Z}_2)^n = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$. Then kG is cotriangular via the bicharacter on $(\mathbb{Z}_2)^n$ given by

$$\langle g_i \mid g_j \rangle = \begin{cases} 1 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$$

extended linearly to kG. Give H a G-grading by setting $\deg(T_i^j) = g_i$ and consider H as a G-Lie coloralgebra via $[\ ,\ ]_G$ defined using the bicharacter. Using the relations in Example 4.3 it is straightforward to check that the two brackets coincide.

Example 4.5: Consider H as in the previous example and V an H-comodule with basis $\{X_1, \ldots, X_n\}$. The H-symmetric algebra (also called the quantum plane) and the H-exterior algebra of V are defined to be, respectively:

$$S_B(V) = T(V)/(\mu \circ (\mathrm{id} - \tau)(X_i \otimes X_j),$$
$$E_B(V) = T(V)/\mu \circ (\mathrm{id} + \tau)(X_i \otimes X_j).$$

Then $S_B(V)$ is H-commutative [CW], hence its associated H-Lie algebra is trivial. As for $E_B(V) = k\langle X_1, \ldots, X_n \mid X_iX_j = -X_jX_i \quad \forall i \neq j\rangle$, it does have a non-trivial H-Lie structure as follows:

$$\begin{aligned}[X_i, X_j] =& X_iX_j - \sum_{k,\ell} \langle T_i^k \mid T_j^\ell \rangle X_jX_i \\ =& X_iX_j - \langle T_i^i \mid T_j^j \rangle X_jX_i \\ =& 2X_iX_j.\end{aligned}$$

Here, as in the previous example, $E_B(V)$ is not a Lie superalgebra but is a Lie coloralgebra for $G = (\mathbb{Z}_2)^n$.

Note that choosing $C = -B$ instead of B would also satisfy the requirements for $K = \mathcal{O}_C(M_n(k))$ to be a cotriangular bialgebra. The above algebras change roles with respect to C: $S_B(V) \cong E_C(V)$ has a non-trivial K-Lie structure, and $E_B(V) \cong S_C(V)$ has a trivial K-Lie structure.

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